On meromorphic functions defined by a differential system of order 1, II

Tristan Torrelli

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Abstract. Given a nonzero germ h of holomorphic function on $(\mathbf{C}^n, 0)$, we study the condition: *"the ideal $\mathrm{Ann}_{\mathcal{D}}\, 1/h$ is generated by operators of order 1"*. When h defines a generic arrangement of hypersurfaces with an isolated singularity, we show that it is verified if and only if h is weighted homogeneous and -1 is the only integral root of its Bernstein-Sato polynomial. When h is a product, we give a process to test this last condition. Finally, we study some other related conditions.

1 Introduction

Let $h \in \mathcal{O} = \mathbf{C}\{x_1, \ldots, x_n\}$ be a nonzero germ of holomorphic function such that $h(0) = 0$. We denote by $\mathcal{O}[1/h]$ the ring $\mathcal{O}$ localized by the powers of h. Let $\mathcal{D} = \mathcal{O}\langle \partial_1, \ldots, \partial_n \rangle$ be the ring of linear differential operators with holomorphic coefficients and $F_\bullet \mathcal{D}$ its filtration by order. In [28], we study the following condition on h:

$\mathbf{A}(1/h)$: The left ideal $\mathrm{Ann}_{\mathcal{D}}\, 1/h \subset \mathcal{D}$ of operators annihilating $1/h$ is generated by operators of order one.

This property is very natural when one considers sections of $\mathcal{O}[1/h]/\mathcal{O}$ with an algebraic viewpoint, see [26]. On the other hand, it seems to be linked to the topological property $\mathbf{LCT}(h)$: *the de Rham complex $\Omega^\bullet[1/h]$ of meromorphic forms with poles along $h = 0$ is quasi-isomorphic to its subcomplex of logarithmic forms*. In particular, $\mathbf{LCT}(h)$ implies $\mathbf{A}(1/h)$ for free germs [8] (in the sense of K. Saito [20]). The study of this condition $\mathbf{LCT}(h)$ was initiated in [9] by F.J. Castro Jiménez, D. Mond and L. Narváez Macarro (see

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[29] for a survey). In this paper, we pursue the study of the condition $\mathbf{A}(1/h)$, and more precisely when h is a reducible germ. Our motivation is to deepen the link between $\mathbf{LCT}(h)$ and $\mathbf{A}(1/h)$.

Let us recall that this last condition is closely linked to the following ones:

$\mathbf{H}(h)$ **:** The germ h belongs to the ideal of its partial derivatives.

$\mathbf{B}(h)$ **:** -1 is the smallest integral root of the Bernstein polynomial of h.

$\mathbf{A}(h)$ **:** The ideal $\mathrm{Ann}_{\mathcal{D}}\, h^s$ is generated by operators of order one.

Indeed, condition $\mathbf{H}(h)$ seems to be necessary in order to have $\mathbf{A}(1/h)$, see [29]. Moreover, condition $\mathbf{A}(1/h)$ always implies $\mathbf{B}(h)$ ([28], Proposition 1.3). This last condition has the following algebraic meaning: *the* $\mathcal{D}$*-module* $\mathcal{O}[1/h]$ *is generated by* $1/h$ (see below). On the other hand, one can easily check that:

If conditions $\mathbf{H}(h)$, $\mathbf{B}(h)$ *and* $\mathbf{A}(h)$ *are verified, then so is* $\mathbf{A}(1/h)$. (1)

Our first part is devoted to condition $\mathbf{B}(h)$. For testing this condition, it seems natural to avoid the full determination of the Bernstein polynomial of h, denoted by $b(h^s, s)$. Here we give such a trick when h is not irreducible, using Bernstein polynomials associated with sections of holonomic $\mathcal{D}$-modules.

Given a nonzero germ $f \in \mathcal{O}$ and an element $m \in \mathcal{M}$ of a holonomic $\mathcal{D}$-module without f-torsion, we recall that there exists a functional equation:

$$b(s)mf^s = P(s) \cdot mf^{s+1} \tag{2}$$

in $(\mathcal{D}m) \otimes \mathcal{O}[1/f, s]f^s$, where $P(s) \in \mathcal{D}[s] = \mathcal{D} \otimes \mathbf{C}[s]$ and $b(s) \in \mathbf{C}[s]$ are nonzero [17]. The *Bernstein polynomial* of f associated with m, denoted by $b(mf^s, s)$, is the monic polynomial $b(s) \in \mathbf{C}[s]$ of smallest degree which verifies such an equation. When f is not a unit and $m \in f^{r-1}\mathcal{M} - f^r\mathcal{M}$ with $r \in \mathbf{N}^*$, it is easy to check that $-r$ is a root of $b(mf^s, s)$. Thus we consider the following condition:

$\mathbf{B}(m, f)$ **:** -1 is the smallest integral root of $b(mf^s, s)$

for $m \in \mathcal{M} - f\mathcal{M}$; this extends our previous notation when $m = 1 \in \mathcal{O} = \mathcal{M}$. By generalizing a well known result due to M. Kashiwara, this condition means: *the* $\mathcal{D}$*-module* $(\mathcal{D}m)[1/f]$ *is generated by* m/f (see Proposition 2.5). Hence we get:

PROPOSITION 1.1 *Let* $h_1, h_2 \in \mathcal{O}$ *be two nonzero germs without common factor and such that* $h_1(0) = h_2(0) = 0$.
(i) We have: $\mathbf{B}(h_1h_2) \Rightarrow \mathbf{B}(1/h_1, h_2) \Rightarrow \mathbf{B}(\dot{1}/h_1, h_2)$ *where* $\dot{1}/h_1 \in \mathcal{O}[1/h_1]/\mathcal{O}$.
(ii) If $\mathbf{B}(h_1)$ *is verified, then* $\mathbf{B}(h_1h_2) \Leftrightarrow \mathbf{B}(1/h_1, h_2)$.
(iii) If $\mathbf{B}(h_2)$ *is verified, then* $\mathbf{B}(1/h_1, h_2) \Leftrightarrow \mathbf{B}(\dot{1}/h_1, h_2)$.

Of course, the equivalence in (ii) just means: $(\mathcal{O}[1/h_1])[1/h_2] = \mathcal{O}[1/h_1h_2]$. Let us insist on the condition $\mathbf{B}(\dot{1}/h_1, h_2)$. Indeed, the polynomial $b((\dot{1}/h_1)h_2^s, s)$ may be considered as a Bernstein polynomial of the function h_2 in restriction to the hypersurface $(X_1, 0) \subset (\mathbf{C}^n, 0)$ defined by h_1, see [26]. In particular, $b((\dot{1}/h_1)h_2^s, s)$ coincides with the (classical) Bernstein Sato polynomial of $h_2|_{X_1} : (X_1, 0) \to (\mathbf{C}, 0)$ if h_1 defines a smooth germ $(X_1, 0)$ (Corollary 2.4); thus this trick is very relevant when h has smooth components. As an application, we prove that $\mathbf{B}(h)$ is true when h defines a hyperplane arrangement (Proposition 2.7), by using the classical principle of 'Deletion-Restriction'. This result was first obtained by A. Leykin [31], and more recently by M. Saito [22].

What about the condition $\mathbf{A}(1/h)$ when $h = h_1 \cdot h_2$ is a product with $h_1(0) = h_2(0) = 0$ and h_1, h_2 have no common factor ? It is also natural to consider the ideal $\mathrm{Ann}_{\mathcal{D}}\,(1/h_1)h_2^s$ and the Bernstein polynomial $b((1/h_1)h_2^s, s)$. Indeed $\mathbf{B}(1/h_1, h_2)$ is a weaker condition than $\mathbf{B}(h_1h_2)$ (Proposition 1.1) and we have an analogue of (1). Of course, it is difficult to verify if $\mathrm{Ann}_{\mathcal{D}}\,(1/h_1)h_2^s$ is - or not - generated by operators of order one. Meanwhile, this may be done under strong assumptions on the components of h, by using the characteristic variety of $\mathcal{D}(1/h_1)h_2^s$ which may be explicited in terms of the one of $\mathcal{D}(1/h_1)$ [14]. Let us give a definition.

Definition 1.2 *A reduced germ $h \in \mathcal{O}$ defines a* generic arrangement of hypersurfaces with an isolated singularity *if it is a product $\prod_{i=1}^p h_i$, $p \geq 2$, of germs h_i which defines an isolated singularity, and such that, for any index $2 \leq k \leq \min(p, n)$, the morphism $(h_{i_1}, \ldots, h_{i_k}) : (\mathbf{C}^n, 0) \to (\mathbf{C}^k, 0)$ defines a complete intersection with an isolated singularity at the origin.*

In the second part, we give a full characterization of $\mathbf{A}(1/h)$ for such a type of germ.

Theorem 1.3 *Let $h = \prod_{i=1}^p h_i \in \mathcal{O}$, $p \geq 2$, define a generic arrangement of hypersurfaces with an isolated singularity. Then the ideal $\mathrm{Ann}_{\mathcal{D}}\,1/h$ is generated by operators of order one if and only if the following conditions are verified:*

1. *the germ h is weighted homogeneous;*
2. *-1 is the only integral root of the Bernstein polynomial of h.*

We recall that a nonzero germ h is *weighted homogeneous* of weight $d \in \mathbf{Q}^+$ for a system $\alpha \in (\mathbf{Q}^{*+})^n$ if there exists a system of coordinates in which h is a linear combination of monomials $x_1^{\gamma_1} \cdots x_n^{\gamma_n}$ with $\sum_{i=1}^n \alpha_i\gamma_i = d$.

This result generalizes the case of a hypersurface with an isolated singularity [26]. Moreover, the condition $\mathbf{B}(h)$ is also explicit when $p=2$, h weighted homogeneous (Corollary 3.6), and the trick above for testing $\mathbf{B}(h)$ may be generalized for $p \geq 3$ (Proposition 2.8). On the other hand, these conditions on the components of h are strong and they are not verified in general. To illustrate this limitation, we end this part by studying the condition $\mathbf{A}(1/h)$ for $h=(x_1-x_2x_3)g$ when $g \in \mathbf{C}[x_1,x_2]$ is a weighted homogeneous polynomial.

PROPOSITION 1.4 *Let $g \in \mathbf{C}[x_1,x_2]$ be a weighted homogeneous reduced polynomial of multiplicity greater or equal to* 3*. Let $h \in \mathbf{C}[x_1,x_2,x_3]$ be the polynomial $(x_1-x_2x_3)g$.*
(i) If g is not homogeneous, then the condition $\mathbf{A}(1/h)$ does not hold for h.
(ii) If g is homogeneous of degree 3*, then $\mathbf{A}(1/h)$ holds for h.*

Here $\mathbf{H}(h)$ are $\mathbf{B}(h)$ are verified (see Lemma 3.7) whereas $\mathbf{A}(h)$ fails. We mention that this family of surfaces was intensively studied by the Sevilian group in order to understand the condition $\mathbf{LCT}(h)$ [4], [6], [10], [12], [13].

In the last part, we give some results on conditions closely linked to $\mathbf{A}(1/h)$. First, we show how the Sebastiani-Thom process allows to construct germs h which verify the condition $\mathbf{A}(h)$. Then, we do some remarks on a natural generalization of condition $\mathbf{A}(1/h)$. We end this note with some remarks on the holonomy of a particular $\mathcal{D}$-module which appears in the study of $\mathbf{LCT}(h)$.

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2 The condition $\mathbf{B}(h)$ for reducible germs

2.1 Preliminaries

In this paragraph, we recall some results about Bernstein polynomials of a germ $f \in \mathcal{O}$ associated with a section m of a holonomic $\mathcal{D}$-module $\mathcal{M}$ without f-torsion. As they appear in [24] (unpublished), we recall some proofs for the convenience of the reader.

LEMMA 2.1 *Let $f \in \mathcal{O}$ be a nonzero germ such that $f(0)=0$. Let m be a germ of holonomic $\mathcal{D}$-module $\mathcal{M}$ without f-torsion. Let $P(s) \in \mathcal{D}[s]$ be a differential operator such that $P(j)mf^j \in \mathcal{M}[1/f]$ is zero for a infinite sequence of integers*

$j \in \mathbf{Z}$. Then $P(s)$ belongs to the annihilator in $\mathcal{D}[s]$ of $mf^s \in \mathcal{M}[1/f,s]f^s$, denoted by $\mathrm{Ann}_{\mathcal{D}[s]}\, mf^s$.

Proof. We have the following identity:

$$P(s)mf^s = (\sum_{i=0}^{d} m_i s^i) f^{s-N} \tag{3}$$

in $\mathcal{M}[1/f,s]f^s$, where $m_i \in \mathcal{M}$ and $N \in \mathbf{N}$ denotes the order of P. By assumption, there exists some integers $j_0 < \cdots < j_d$ such that $\sum_{i=0}^{d} (j_k)^i m_i = 0$ in $\mathcal{M}$ for $0 \leq k \leq d$. Since the Gram matrix of the integers $j_0, \ldots, j_d$ is inversible, the previous identities imply that $m_i = 0$ for $0 \leq i \leq d$. We conclude with (3). $\square$

LEMMA 2.2 *Let $f \in \mathcal{O}$ be a nonzero germ such that $f(0) = 0$. Let $m \in \mathcal{M}$ be a nonzero section of a holonomic $\mathcal{D}$-module without f-torsion.*
(i) If $g \in \mathcal{O}$ is such that $g \cdot m = 0$, then $b(mf^s, s)$ coincides with $b(m(f+g)^s, s)$.
(ii) If $m \in \mathcal{M} - f\mathcal{M}$, then $(s+1)$ divides $b(mf^s, s)$.
(iii) For all $p \in \mathbf{N}^*$, $b(mf^{ps}, s)$ divides the $\prod_{i=0}^{p-1} b(mf^s, ps+i)$, and the polynomial l.c.m$(b(mf^s, ps), \ldots, b(mf^s, ps+p-1))$ divides $b(mf^{ps}, s)$. In particular, these polynomials have the same roots.

Proof. In order to prove the first point, we just have to check that the polynomial $b(m(f+g)^s, s)$ is a multiple of $b(mf^s, s)$ for any $g \in \mathrm{Ann}_{\mathcal{O}}\, m$, and to apply this fact with $\tilde{f} = f + g$, $\tilde{g} = -g$. Let $P(s) \in \mathcal{D}[s]$ be a differential operator which realizes the Bernstein polynomial of $m(f+g)^s$. In particular, $R(s) = b(m(f+g)^s, s) - P(s)f$ belongs to $\mathrm{Ann}_{\mathcal{D}[s]}\, m(f+g)^s$. As $(f+g)^j \cdot m = f^j \cdot m$ for all $j \in \mathbf{N}$, the operator $R(s)$ annihilates mf^s by Lemma 2.1. Thus the polynomial $b(mf^s, s)$ divides $b(m(f+g)^s, s)$.

Now, we prove (ii). Let $R \in \mathcal{D}$ be the remainder in the division of $P(s)$ by $(s+1)$ in a nontrivial identity (2). Thus $R \cdot mf^{s+1} = (R \cdot m)f^{s+1} + (s+1)af^s$ where $a \in \mathcal{M}[1/f, s]$. From (2), we get $b(-1)m = fR(m)$. Hence $b(-1) = 0$ since $m \notin f\mathcal{M}$.

The last point is an easy exercice. $\square$

PROPOSITION 2.3 *Let $X \subset \mathbf{C}^n$ be an analytic subvariety of codimension p passing through the origin. Let $i : X \hookrightarrow \mathbf{C}^n$ denote the inclusion and let $h_1, \ldots, h_p \in \mathcal{O}$ be local equations of $i(X)$. Let $f \in \mathcal{O}$ be a germ such that $f \circ i$ is not constant and let $\mathcal{M}'$ be a holonomic $\mathcal{D}_{X,0}$-module without $(f \circ i)$-torsion.*

If $m \in \mathcal{M}'$ is nonzero, then $b(m(f \circ i)^s, s)$ coincides with the polynomial $b(i_+(m)f^s, s)$ where $i_+(m) \in \mathcal{M}' \otimes (\mathcal{O}[1/h_1 \cdots h_p] / \sum_{i=1}^{p} \mathcal{O}[1/h_1 \cdots \check{h}_i \cdots h_p])$ denotes the element $\dot{1}/h_1 \cdots h_p$.

Proof. Up to a change of coordinates, we can assume that $h_i = x_i$, $1 \le i \le p$. Then the remainder $\tilde{f} \in \mathbf{C}\{x_{p+1}, \ldots, x_n\}$ in the division of f by $x_1, \ldots, x_p$ defines the germ $f \circ i$. Thus we have $b(i_+(m)f^s, s) = b(i_+(m)\tilde{f}^s, s)$ by using Lemma 2.2. Let us prove that $b(i_+(m)\tilde{f}^s, s)$ coincides with $b(m\tilde{f}^s, s)$. Firstly, it easy to check that a functional equation for $b(m\tilde{f}^s, s)$ induces an equation for $b(i_+(m)\tilde{f}^s, s)$; thus $b(i_+(m)\tilde{f}^s, s)$ divides $b(m\tilde{f}^s, s)$. On the other hand, we consider the following equation:

$$b(i_+(m)\tilde{f}^s, s) i_+(m)\tilde{f}^s = P \cdot i_+(m)\tilde{f}^{s+1} \tag{4}$$

where $P \in \mathcal{D}[s]$. It may be written $P = \sum_{i=1}^{p} Q_i x_i + R$ where $Q_i \in \mathcal{D}[s]$ and the coefficients of $R \in \mathcal{D}[s]$ do not depend on $x_1, \ldots, x_p$; in particular, we can change P by R in (4). Let $\tilde{R} \in \mathcal{D}_{X,0}[s] = \mathbf{C}\{x_{p+1}, \ldots, x_n\}\langle \partial_{p+1}, \ldots, \partial_n \rangle [s]$ denote the constant term of R as an operator in $\partial_1, \ldots, \partial_p$ with coefficients in $\mathcal{D}_{X,0}[s]$. Obviously we can change R by $\tilde{R}$ in (4). As the annihilator of $i_+(m)\tilde{f}^s$ in $\mathcal{D}_{X,0}[s]$ coincides with the one of $m\tilde{f}^s$, we deduce that $b(i_+(m)\tilde{f}^s, s)$ is a multiple of $b(m\tilde{f}^s, s)$. This completes the proof. $\square$

Corollary 2.4 *Let $h_1, h_2 \in \mathcal{O}$ be two nonzero germs without common factor and such that $h_1(0) = h_2(0) = 0$. Assume that h_1 defines a smooth germ $(X_1, 0) \subset (\mathbf{C}^n, 0)$. Then $b((1/h_1)h_2^s, s)$ coincides with the (classical) Bernstein Sato polynomial of $h_2|_{X_1} : (X_1, 0) \to (\mathbf{C}, 0)$.*

Proposition 2.5 *Let $f \in \mathcal{O}$ be a nonzero germ such that $f(0) = 0$. Let m be a section of a holonomic $\mathcal{D}$-module without f-torsion, and $\ell \in \mathbf{N}^*$. The following conditions are equivalent:*

1. *The smallest integral root of $b(mf^s, s)$ is strictly greater than $-\ell - 1$.*
2. *The $\mathcal{D}$-module $(\mathcal{D}m)[1/f]$ is generated by $mf^{-\ell}$.*
3. *The following morphism is an isomorphism:*

$$\begin{array}{ccc} \dfrac{\mathcal{D}[s]mf^s}{(s+\ell)\mathcal{D}[s]mf^s} & \longrightarrow & (\mathcal{D}m)[1/f] \\ P(s)mf^s & \mapsto & P(-\ell) \cdot mf^{-\ell} \ . \end{array}$$

This is a direct generalization of a well known result due to M. Kashiwara and J.E. Björk for $m = 1 \in \mathcal{O} = \mathcal{M}$ (see [16] Proposition 6.2, [2] Propositions 6.1.18, 6.3.15 & 6.3.16).

2.2 Is -1 the only integral root of $b(h^s, s)$?

First of all, let us prove Proposition 1.1.

Proof of Proposition 1.1. Assume that condition $\mathbf{B}(h_1h_2)$ is verified. From Proposition 2.5, this means $\mathcal{D}1/h_1h_2 = \mathcal{O}[1/h_1h_2]$. In particular, we have $(\mathcal{D}1/h_1)[1/h_2] \subset \mathcal{D}1/h_1h_2$; thus, by using Proposition 2.5 with $m = 1/h_1$, condition $\mathbf{B}(1/h_1, h_2)$ is verified. The second relation in (i) is clear since a functional equation realizing $b((1/h_1)h_2^s, s)$ induces a functional equation for $b((\dot{1}/h_1)h_2^s, s)$.

The second point is clear, since it just means $(\mathcal{O}[1/h_1])[1/h_2] = \mathcal{O}[1/h_1h_2]$ (using three times Proposition 2.5). Now, given $P \in \mathcal{D}$ and $\ell \in \mathbf{N}$, let us prove that $(P \cdot 1/h_1) \otimes 1/h_2^\ell$ belongs to $\mathcal{D}1/h_1h_2$ when $\mathbf{B}(\dot{1}/h_1, h_2)$ and $\mathbf{B}(h_2)$ are verified. From Proposition 2.5, there exists an operator $Q \in \mathcal{D}$ such that $(P \cdot \dot{1}/h_1) \otimes 1/h_2^\ell = Q \cdot \dot{1}/h_1 \otimes 1/h_2$ in $(\mathcal{O}[1/h_1]/\mathcal{O})[1/h_2]$. Hence we have $(P \cdot 1/h_1) \otimes 1/h_2^\ell = Q \cdot 1/h_1h_2 + a/h_2^N$, where $a \in \mathcal{O}$ and $N \in \mathbf{N}^*$. As condition $\mathbf{B}(h_2)$ is verified, there exists $R \in \mathcal{D}$ such that $R \cdot 1/h_2 = a/h_2^N$. Thus we get $(P \cdot 1/h_1) \otimes 1/h_2^\ell = (Q + Rh_1) \cdot 1/h_1h_2$. In consequence, the condition $\mathbf{B}(1/h_1, h_2)$ is also verified. □

The following examples show that there is no other relation between $\mathbf{B}(h_1h_2)$, $\mathbf{B}(1/h_1, h_2)$, $\mathbf{B}(\dot{1}/h_1, h_2)$ and $\mathbf{B}(h_1)$, $\mathbf{B}(h_2)$.

Example 2.6 (i) If $h_1 = x_1$ and $h_2 = x_1 + x_2x_3 + x_4x_5$, then $b(h_1^s, s) = b(h_2^s, s) = s + 1$ but $b((\dot{1}/h_1)h_2^s, s) = b((x_2x_3 + x_4x_5)^s, s) = (s + 1)(s + 2)$ by using Corollary 2.4.

(ii) If $h_1 = x_1x_2 + x_3x_4$ and $h_2 = x_1x_2 + x_3x_5$, then $b(h_1^s, s) = b(h_2^s, s) = (s+1)(s+2)$, but $b((h_1h_2)^s, s)$ is equal to $(s+1)^4(s+3/2)^2$ by using Macaulay 2 [15], [18]. Moreover, if $h_3 = x_1$, then condition $\mathbf{B}(h_1h_3)$ is also true, since $b((h_1h_3)^s, s) = (s + 1)^3(s + 3/2)$ using Macaulay 2. Hence condition $\mathbf{B}(h_1h_2)$ does not depend in general of the conditions $\mathbf{B}(h_1)$ and $\mathbf{B}(h_2)$.

(iii) Assume that $h_1 = x_1$ and $h_2 = x_1^2 + x_2^4 + x_3^4$. Then $b(h_1^s, s) = s + 1$ and condition $\mathbf{B}(\dot{1}/h_1, h_2)$ is true, since $b((\dot{1}/h_1)h_2^s, s) = b((x_2^4+x_3^4)^s, s)$ by Corollary 2.4. But a direct computation using [25] shows that condition $\mathbf{B}(1/h_1, h_2)$ is false.

(iv) Assume that $h_1 = x_1x_2x_3 + x_4x_5$ and $h_2 = x_1$. Then $b((1/h_1)h_2^s, s) = b((\dot{1}/h_1)h_2^s, s) = b((x_4x_5)^s, s) = (s + 1)^2$, using [27] Proposition 2.9 and [25] Proposition 1. On the other hand, $(s + 1)(s + 2)$ divides $b((h_1h_2)^s, s)$ and $b(h_1^s, s)$, by the semi-continuity of the Bernstein polynomial (since when u is a unit, we have $b((u(ux_2x_3 + x_4x_5))^s, s) = (s + 1)(s + 2)$). Thus $\mathbf{B}(1/h_1, h_2)$ does not imply $\mathbf{B}(h_1h_2)$ in general.

As an application of Proposition 1.1, we obtain a new proof of the following result.

PROPOSITION 2.7 ([31], [22]) *Let $h \in \mathbf{C}[x_1, \ldots, x_n]$ be the product of nonzero linear forms (distinct or not). Then the Bernstein polynomial of h has only -1 as integral root.*

Proof. Let h be the product $l_1^{p_1} \cdots l_r^{p_r}$ where $r, p_1, \ldots, p_r \in \mathbf{N}^*$ are positive integers, and $l_i \in (\mathbf{C}^n)^\star$ are distinct. We prove the result by induction on r. If $r = 1$, this is a direct consequence of the following identity:

$$\frac{1}{p^p}\left(\frac{\partial}{\partial x}\right)^p \cdot (x^p)^{s+1} = (s+\frac{1}{p})(s+\frac{2}{p})\cdots(s+\frac{p-1}{p})(s+1)(x^p)^s$$

for $p \in \mathbf{N}^*$. Now, we assume that the assertion is true for any germ as above with at most $N \geq 1$ distinct irreducible components. Let h be such a germ with $r = N$. Let $l \in (\mathbf{C}^n)^\star$ be a nonzero form which is not a factor of h, and $p \in \mathbf{N}^*$. In particular, -1 is the only integral root of the Bernstein polynomial of l, l^p and h. Let us remark that the assertion for $h \cdot l$ implies the assertion for $h \cdot l^p$. Indeed, using Lemma 2.2, it is easy to check that $\mathbf{B}(1/h, l)$ implies $\mathbf{B}(1/h, l^p)$. We conclude with the help of Proposition 1.1, (ii).

In order to prove $\mathbf{B}(h \cdot l)$, we just have to check that -1 is the only integral root of $b((\dot{1}/l)h^s, s)$ (Proposition 1.1, (iii)). But this is true by induction on N since this last polynomial coincides with the Bernstein polynomial of $h|_{\{l=0\}}$ (Corollary 2.4). This completes the proof. $\square$

When h has more than two components, the following result provides a generalized criterion for the condition $\mathbf{B}(h)$.

PROPOSITION 2.8 *Let $h_1, \ldots, h_p \in \mathcal{O}$ be nonzero germs without common factor, and such that $h_1(0) = \cdots = h_p(0) = 0$.*

(i) Assume that $2 \leq p \leq n$ and that $(h_1, \ldots, h_p)$ defines a complete intersection. If $\mathbf{B}(h_1 \cdots \check{h}_j \cdots h_p)$, $1 \leq j \leq p$, are verified, then $\mathbf{B}(\delta, h_1)$ implies $\mathbf{B}(h_1 \cdots h_p)$ where $\delta = \dot{1}/h_2 \cdots h_p \in \mathcal{O}[1/h_2 \cdots h_p]/\sum_{i=2}^p \mathcal{O}[1/h_2 \cdots \check{h}_i \cdots h_p]$.

(ii) Assume that $p = n$ and $(h_1, \ldots, h_n)$ defines the origin. If the conditions $\mathbf{B}(h_1 \cdots \check{h}_j \cdots h_n)$, $1 \leq j \leq n$, are verified, then so is $\mathbf{B}(h_1 \cdots h_n)$.

(iii) Assume that $p \geq n+1$. If the conditions $\mathbf{B}(h_{i_1} \cdots h_{i_n})$ are verified for $1 \leq i_1 < \cdots < i_n \leq p$ then so is $\mathbf{B}(h_1 \cdots h_p)$.

Proof. We start with the first assertion. From Proposition 1.1, we just have to prove $\mathbf{B}(1/h_2 \cdots h_p, h_1)$ (since $\mathbf{B}(h_2 \cdots h_p)$ is verified). Thus, given $P \in \mathcal{D}$ and $\ell \in \mathbf{N}$, let us prove that $(P \cdot 1/h_2 \cdots h_p) \otimes 1/h_1^\ell$ belongs to $\mathcal{D}1/h_1 \cdots h_p$. Using condition $\mathbf{B}(\delta, h_1)$, we have

$$(P \cdot \frac{1}{h_2 \cdots h_p}) \otimes \frac{1}{h_1^\ell} = R \cdot \frac{1}{h_1 \cdots h_p} + \sum_{2 \leq i \leq p} \frac{q_i}{h_1^{\ell_{i,1}} \cdots \check{h}_i^{\ell_{i,i}} \cdots h_p^{\ell_{i,p}}}$$

with $q_i \in \mathcal{O}$ and $\ell_{i,j} \in \mathbf{N}$. We conclude by using that $\mathcal{O}[1/h_1 \cdots \check{h}_i \cdots h_p]$ is generated by $1/h_1 \cdots \check{h}_i \cdots h_p$ for $2 \leq i \leq p$ by assumption.

In order to prove (ii), we have to check that $\mathbf{B}(\delta, h_1)$ is verified when $p = n$. Firstly, we notice that the $\mathcal{D}$-module $\mathcal{O}[1/h_2 \cdots h_p]/\sum_{i=2}^{p} \mathcal{O}[1/h_2 \cdots \check{h}_i \cdots h_p]$ is generated by δ (using condition $\mathbf{B}(h_2 \cdots h_p)$). Thus $\mathcal{N} = (\mathcal{D}\delta)[1/h_1]/\mathcal{D}\delta$ is isomorphic to the module of local algebraic cohomology with support in the origin; in particular, any nonzero section generates $\mathcal{N}$. We deduce easily that $(\mathcal{D}\delta)[1/h_1]$ is generated by $\delta \otimes 1/h_1$. From Proposition 2.5, the condition $\mathbf{B}(\delta, h_1)$ is verified.

The last point is a direct consequence of the following fact, proved by A. Leykin [31], Remark 5.2: *if the condition* $\mathbf{B}(h_{i_1} \cdots h_{i_{k-1}})$ *is verified for* $1 \leq i_1 < \cdots < i_{k-1} \leq k$ *with* $k \geq n+1$, *then so is* $\mathbf{B}(h_1 \cdots h_k)$. $\square$

Example 2.9 Let $n = 3$, $p \geq 3$ and $h_i = a_{i,1}x_1^2 + a_{i,2}x_2^3 + a_{i,3}x_3^4$ where the vector $a_i = (a_{i,1}, a_{i,2}, a_{i,3})$ belongs to $\mathbf{C}^3$ and the rank of $(a_{i_1}, a_{i_2}, a_{i_3})$ is maximal for $1 \leq i_1 < i_2 < i_3 \leq p$. Thus the polynomial $h = h_1 \cdots h_p$ defines a generic arrangement of hypersurfaces with an isolated singularity. By using the closed formulas for $b(h_i^s, s)$ and $b((1/h_i)h_j^s, s)$, $1 \leq i \neq j \leq p$, (see [32], [25]), it is easy to check that the conditions $\mathbf{B}(h_i)$ and $\mathbf{B}(1/h_i, h_j)$ are verified; thus so is $\mathbf{B}(h)$.

3 The condition $\mathbf{A}(1/h)$ for a generic arrangement of hypersurfaces with an isolated singularity

In this part, we characterize the condition $\mathbf{A}(1/h)$ when $h \in \mathcal{O}$ defines a generic arrangement of hypersurfaces with an isolated singularity. Then we study this condition for a particular family of free germs (§3.3).

3.1 A convenient annihilator

This paragraph is devoted to the determination of an annihilator which will allow us to characterize $\mathbf{A}(1/h)$.

Notation 3.1 *Let* $h = (h_1, \ldots, h_r) : \mathbf{C}^n \to \mathbf{C}^r$, $1 \leq r < n$, *be an analytic morphism. For any* $K = (k_1, \ldots, k_{r+1}) \in \mathbf{N}^{r+1}$ *where* $1 \leq k_1, \ldots, k_{r+1} \leq n$ *and* $k_i \neq k_j$ *for* $i \neq j$, *let* $\Delta_K^h \in \mathcal{D}$ *denote the vector field:*

$$\sum_{i=1}^{r+1} (-1)^i m_{K(i)}(h) \partial_{k_i} = \sum_{i=1}^{r+1} (-1)^i \partial_{k_i} m_{K(i)}(h)$$

where $K(i) = (k_1, \ldots, \check{k}_i, \ldots, k_{r+1}) \in \mathbf{N}^r$ *and* $m_{K(i)}(h)$ *is the determinant of the* $r \times r$ *matrix obtained from the Jacobian matrix of* h *by deleting the* k*-th columns with* $k \notin \{k_1, \ldots, \check{k}_i, \ldots, k_{r+1}\}$*.*

Proposition 3.2 *Assume that* $n \geq 3$*. Let* $h = \prod_{i=1}^{p} h_i \in \mathcal{O}$*,* $p \geq 2$*, define a generic arrangement of hypersurfaces with an isolated singularity, and let* $\tilde{h}$ *be the product* $\prod_{i=2}^{p} h_i$*. Then the ideal* $\mathrm{Ann}_{\mathcal{D}}\,(1/\tilde{h})h_1^s$ *is generated by the operators:*

$$\Delta_K^{h_{i_1},\ldots,h_{i_r}} \prod_{i \neq i_1,\ldots,i_r} h_i$$

with $1 \leq r \leq \min(n-1, p)$ *and* $1 = i_1 < \cdots < i_r \leq p$*.*

Proof. Let $I \subset \mathcal{D}$ be the left ideal generated by the given operators, and let $\mathcal{I} \subset \mathcal{O}[\xi_1, \ldots, \xi_n]$ denote the ideal generated by their principal symbols. We will just prove that $\mathrm{Ann}_{\mathcal{D}}\,(1/\tilde{h})h_1^s \subset I$, since the reverse inclusion is obvious. Let us study $\mathrm{char}_{\mathcal{D}}\,\mathcal{D}(1/\tilde{h})h_1^s \subset T^*\mathbf{C}^n$ the characteristic variety of $\mathcal{D}(1/\tilde{h})h_1^s$. Given an analytic subspace $X \subset \mathbf{C}^n$, we denote by $W_{h_1|X}$ the closure in $T^*\mathbf{C}^n$ of the set $\{(x, \xi + \lambda dh_1(x)) \mid \lambda \in \mathbf{C},\, (x,\xi) \in T_X^*\mathbf{C}^n\}$.

Assertion 1. The characteristic variety of $\mathcal{D}(1/\tilde{h})h_1^s$ *is the union of the subspaces* W_{h_1} *and* $W_{h_1|X_{i_1,\ldots,i_r}}$*,* $2 \leq i_1 < \cdots < i_r \leq p$*,* $1 \leq r \leq \min(n-1,p)$*, where* $X_{i_1,\ldots,i_r} \subset \mathbf{C}^n$ *is the complete intersection defined by* $h_{i_1}, \ldots, h_{i_r}$*.*

Proof. Under our assumption, $(\tilde{h}^{-1}(0), x)$ is a germ of a normal crossing hypersurface for any $x \in \tilde{h}^{-1}(0)/\{0\}$ close enough to the origin. In particular, $\mathcal{D}1/\tilde{h}$ coincides with $\mathcal{O}[1/h_{i_1} \cdots h_{i_r}]$ on a neighborhood of such a point, where $\{i_1, \ldots, i_r\} = \{i \mid h_i(x) = 0,\ 2 \leq i \leq p\}$. Hence, the components of the characteristic variety of $\mathcal{D}1/\tilde{h}$ which are not supported by $h_1 = 0$ are $T^*_{\mathbf{C}^n}\mathbf{C}^n$ and the conormal spaces $T^*_{X_{i_1,\ldots,i_r}}\mathbf{C}^n$, with $2 \leq i_1 < \cdots < i_r \leq p$ and $1 \leq r \leq \min(n-1, p)$. The assertion follows from a result of V. Ginzburg ([14] Proposition 2.14.4). □

We recall that the relative conormal space[2] $W_{h_1} \subset T^*\mathbf{C}^n$ is defined by the polynomials $\sigma(\Delta_{k_1,k_2}^{h_1}) = h'_{1,x_{k_2}}\xi_{k_1} - h'_{1,x_{k_1}}\xi_{k_2}$, $1 \leq k_1 < k_2 \leq n$ (see [32] for example). One can also determine explicitly the defining ideal of the spaces $W_{h_1|X_{i_1,\ldots,i_r}}$.

Assertion 2 ([25]). *The conormal space* $W_{h_1|X_{i_1,\ldots,i_r}}$ *is defined by* $h_{i_1}, \ldots, h_{i_r}$ *and by the principal symbol of the vector fields* $\Delta_K^{h_{i_1},\ldots,h_{i_r}}$ *(when* $r < n-1$*), where* $K = (k_1, \ldots, k_{r+2}) \in \mathbf{N}^{r+2}$ *with* $1 \leq k_1 < \cdots < k_{r+2} \leq n$*.*

Now we can determine the equations of $\mathrm{char}_{\mathcal{D}}\,\mathcal{D}(1/\tilde{h})h_1^s$.

[2] See §4.1

Assertion 3. The defining ideal of $\mathrm{char}_{\mathcal{D}}\, \mathcal{D}(1/\tilde{h})h_1^s$ *is included in* $\mathcal{I}$.

Proof. Let $A \in \mathcal{O}[\xi] = \mathcal{O}[\xi_1, \ldots, \xi_n]$ be a polynomial which is zero on the characteristic variety of $\mathcal{D}(1/\tilde{h})h_1^s$. We will prove the result when $p \geq n$ - the case $p \leq n-1$ is analogous.

Using the inclusion $W_{h_1|X_{i_1,\ldots,i_{n-1}}} \subset \mathrm{char}_{\mathcal{D}}\, \mathcal{D}(1/\tilde{h})h_1^s$ and Assertion 2, we have: $A \in (h_{i_1}, \ldots, h_{i_{n-1}})\mathcal{O}[\xi]$ for $2 \leq i_1 < \cdots < i_{n-1} \leq p$. By an easy induction on $p \geq n$, one can check that:

$$\bigcap_{2\leq i_1<\cdots<i_{n-1}\leq p} (h_{i_1}, \ldots, h_{i_{n-1}})\mathcal{O} = \sum_{2\leq i_1<\cdots<i_{n-2}\leq p} [\prod_{i\neq 1,i_1,\ldots,i_{n-2}} h_i]\mathcal{O}$$

using that every sequence $(h_{i_1}, \ldots, h_{i_n})$ is regular. Thus A may be written as a sum $\sum_{2\leq i_1<\cdots<i_{n-2}\leq p} A^{(0)}_{i_1,\ldots,i_{n-2}}(\prod_{i\neq 1,i_1,\ldots,i_{n-2}} h_i)$ for some $A^{(0)}_{i_1,\ldots,i_{n-2}} \in \mathcal{O}[\xi]$.

Now let us fix $i_1 < \cdots < i_{n-2}$ a family of index as above. From the inclusion $W_{h_1|X_{i_1,\ldots,i_{n-2}}} \subset \mathrm{char}_{\mathcal{D}}\, \mathcal{D}(1/\tilde{h})h_1^s$ and Assertion 2, A belongs to the ideal $\mathcal{I}_{1,i_1,\ldots,i_{n-2}} = (h_{i_1}, \ldots, h_{i_{n-2}})\mathcal{O}[\xi] + \sum_K \sigma(\Delta_K^{h_1,h_{i_1},\ldots,h_{i_{n-2}}})\mathcal{O}[\xi]$. On the other hand, let us remark that h_i is $\mathcal{O}[\xi]/\mathcal{I}_{1,i_1,\ldots,i_{n-2}}$-regular for $i \neq 1, i_1, \ldots, i_{n-2}$ [by the principal ideal theorem, using that $\mathcal{I}_{1,i_1,\ldots,i_{n-2}}$ defines the irreducible space $W_{h_1|X_{1,i_1,\ldots,i_{n-2}}}$ of pure dimension $n+1$]. Thus we have $A^{(0)}_{i_1,\ldots,i_{n-2}} \in \mathcal{I}_{1,i_1,\ldots,i_{n-2}}$, and A may be written: $A = U + \sum_{2\leq i_1<\cdots<i_{n-3}\leq p} A^{(1)}_{i_1,\ldots,i_{n-3}}(\prod_{i\neq 1,i_1,\ldots,i_{n-3}} h_i)$ where $A^{(1)}_{i_1,\ldots,i_{n-3}} \in \mathcal{O}[\xi]$ and $U \in \mathcal{I}$. Up to a division by $\mathcal{I}$, we can assume that $U = 0$. After iterating this process with $W_{h_1|X_{i_1,\ldots,i_r}}$, $1 \leq r \leq n-2$, we deduce that $A - A^{(n-2)}\tilde{h}$ belongs to $\mathcal{I}$. Hence, using that $W_{h_1} \subset \mathrm{char}_{\mathcal{D}}\, \mathcal{D}(1/\tilde{h})h_1^s$, we have: $A^{(n-2)} \in \sum_{1\leq k_1<k_2\leq n} \sigma(\Delta^{h_1}_{k_1,k_2})\mathcal{O}[\xi]$. In particular, $A^{(n-2)}\tilde{h}$ belongs to $\mathcal{I}$, and we conclude that $A \in \mathcal{I}$. $\square$

Now let us prove the proposition. Let $P \in \mathrm{Ann}_{\mathcal{D}}\,(1/\tilde{h})h_1^s$ be a nonzero operator of order d. In particular, $\sigma(P)$ is zero on $\mathrm{char}_{\mathcal{D}}\, \mathcal{D}(1/\tilde{h})h_1^s$, and by Assertion 3: $\sigma(P) \in \mathcal{I}$. In other words, there exists $Q \in I$ such that $\sigma(Q) = \sigma(P)$. Thus, the operator $P - Q \in \mathrm{Ann}_{\mathcal{D}}\,(1/\tilde{h})h_1^s \cap F_{d-1}\mathcal{D}$ belongs to I, and so does P (by induction on the order of operators). $\square$

Remark 3.3 We are not able to determine $\mathrm{Ann}_{\mathcal{D}}\, h^s$ when h defines a generic arrangement of hypersurfaces with an isolated singularity. In particular, we do not know if the condition $\mathbf{A}(h)$ (or $\mathbf{W}(h)$) is - or not - verified (see §4.1).

Given a germ $h \in \mathcal{O}$ such that $h(0) = 0$, let us denote by $\mathrm{Der}(-\log h)$ the coherent $\mathcal{O}$-module of logarithmic derivations relative to h, that is, vector fields which preserve $h\mathcal{O}$ (see [19]).

Corollary 3.4 *Let* $h = \prod_{i=1}^p h_i \in \mathcal{O}$, $p \geq 2$, *define a generic arrangement of hypersurfaces with an isolated singularity. Assume that* $n \geq 3$ *and that* h

is a weighted homogeneous polynomial. Then $\mathrm{Der}(-\log h)$ *is generated by the Euler vector field* χ *such that* $\chi(h) = h$ *and the vector fields*

$$\left[\prod_{i\neq i_1,\ldots,i_r} h_i\right] \cdot \Delta_K^{h_{i_1},\ldots,h_{i_r}}$$

where $1 \leq r \leq \min(n-1,p)$ *and* $1 = i_1 < \cdots < i_r \leq p$.

Proof. We denote by $\tilde{h} \in \mathcal{O}$ the product $h_2 \cdots h_p$. Let v be a logarithmic vector field; in particular, $v(h) = ah$. As $h = h_1\tilde{h}$, it is easy to check that $v(h_1) = a_1 h_1$ and $v(\tilde{h}) = \tilde{a}h_1$ for $a_1, \tilde{a} \in \mathcal{O}$ such that $a_1 + \tilde{a} = a$. In particular, $v \cdot (1/\tilde{h})h_1^s = (a_1 s - \tilde{a})(1/\tilde{h})h_1^s$. Thus $v + \tilde{a} - a_1\chi$ belongs to $\mathrm{Ann}_{\mathcal{D}}\,(1/\tilde{h})h_1^s$, and by using the proof of the previous result, we have:

$$v = -\tilde{a} + a_1\chi + \sum_{r=1}^{\min(n-1,p)} \sum_{1\leq i_1<\ldots<i_r\leq p} \lambda_{i_1,\ldots,i_r}\Delta_K^{i_1,\ldots,i_r} \cdot \prod_{i\neq i_1,\ldots,i_r} h_i$$

where $\lambda_{i_1,\ldots,i_r} \in \mathcal{O}$ for $1 \leq i_1 < \ldots < i_r \leq p$. As v is a vector field, we get $v = a_1\chi + \sum_r \sum \lambda_{i_1,\ldots,i_r}[\prod_{i\neq i_1\cdots i_r} h_i]\Delta_K^{i_1,\ldots,i_r}$ and the assertion follows. □

3.2 The expected characterization

The proof of Theorem 1.3 is an easy consequence of the following result

PROPOSITION 3.5 *Let* $h = \prod_{i=1}^p h_i \in \mathcal{O}$, $p \geq 2$, *define a generic arrangement of hypersurfaces with an isolated singularity. Assume that* $n \geq 3$ *and that the origin is a critical point of* h_1. *Let* $\tilde{h}$ *denote the product* $\prod_{i=2}^p h_i$. *Then the ideal* $\mathrm{Ann}_{\mathcal{D}}\,1/h$ *is generated by operators of order one if and only if the following conditions are verified:*

1. *the germ is weighted homogeneous;*

2. -1 *is the smallest integral root of the Bernstein polynomial* $b((1/\tilde{h})h_1^s, s)$.

Proof. We can assume that h does not define a normal crossing divisor. Indeed, the conditions $\mathbf{A}(1/h)$, 1 and 2 are obviously verified for a normal crossing divisor. In particular, the constant term with the coefficient on the right side of any operator in $\mathrm{Ann}_{\mathcal{D}}(1/\tilde{h})h_1^s$ is not a unit (see Proposition 3.2).

Firstly, we prove that conditions 1 & 2 imply $\mathbf{A}(1/h)$. By an Euclidean division, we have a decomposition

$$\mathrm{Ann}_{\mathcal{D}[s]}\,\frac{1}{\tilde{h}}h_1^s = \mathcal{D}[s](s - \tilde{q} - v) + \mathcal{D}[s]\mathrm{Ann}_{\mathcal{D}}\frac{1}{\tilde{h}}h_1^s$$

where v denotes the Euler vector field such that $v(h_1) = h_1$ and $v(\tilde{h}) = \tilde{q}\tilde{h}$ with $\tilde{q} \in \mathbf{Q}^{*+}$. Moreover, with the condition 2, the ideal $\mathrm{Ann}_{\mathcal{D}}\, 1/(\tilde{h}h_1)$ is obtained by fixing $s = -1$ in a system of generators of $\mathrm{Ann}_{\mathcal{D}[s]}(1/\tilde{h})h_1^s$ (see [26] Proposition 3.1). From Proposition 3.2, the condition $\mathbf{A}(1/h)$ is therefore verified.

Now, we prove the reverse. Let us assume that $\mathrm{Ann}_{\mathcal{D}}1/h$ is generated by the operators $Q_1, \ldots, Q_w \in F_1\mathcal{D}$. From Proposition 1.3 in [28], $\mathbf{B}(h)$ is verified, and so[3] is condition 2 by Proposition 1.1. Hence, we just have to check that h is necessarily weighted homogeneous. Let q_i be the germ $Q_i(1) \in \mathcal{O}$ and Q_i' the vector field $Q_i - q_i$. In particular, we have $Q_i'(h) = q_i h$ for $1 \leq i \leq w$. As $h = h_1\tilde{h}$, it is easy to deduce that $Q_i'(\tilde{h}) = \tilde{q}_i\tilde{h}$ and $Q_i'(h_1) = q_{i,1}h_1$ where $\tilde{q}_i, q_{i,1} \in \mathcal{O}$ verify

$$\tilde{q}_i + q_{i,1} = q_i,\ 1 \leq i \leq w.$$

On the other hand, we have the following fact:

Assertion 1. There exists a differential operator R in $\mathrm{Ann}_{\mathcal{D}}(1/\tilde{h})h_1^s$ such that $R = 1 + \sum_{i=1}^{w} A_i q_{i,1}$ with $A_i \in \mathcal{D}$.

Proof. The proof is analogous to the one of [26] Lemme 3.3. From [14] p 351 or [24], there exists a 'good' operator $R_0(s)$ of degree $N \geq 1$ in $\mathrm{Ann}_{\mathcal{D}[s]}(1/\tilde{h})h_1^s$, that is $R_0(s) = s^N + \sum_{k=0}^{N-1} s^k P_k$ with $P_k \in F_{N-k}\mathcal{D}$, $0 \leq k \leq N-1$. By Euclidean division, we have $R_0(s) = (s+1)S(s) + R_0(-1)$ where $S(s)$ is monic in s of degree $N-1$ and $R_0(-1) \in \mathrm{Ann}_{\mathcal{D}}1/h$. Thus, there exists $A_1, \ldots, A_w \in \mathcal{D}$ such that $R_0(-1) = \sum_{i=1}^{w} A_i Q_i$. From the relations above, we get

$$(s+1)S(s)\frac{1}{\tilde{h}}h_1^s + (s+1)\sum_{i=1}^{w} A_i q_{i,1}\frac{1}{\tilde{h}}h^s = 0.$$

Hence $R_1(s) = S(s) + \sum_{i=1}^{w} A_i q_{i,1}$ belongs to $\mathrm{Ann}_{\mathcal{D}[s]}(1/\tilde{h})h_1^s$. By iteration, we can assume that $S(s) = 1$. $\square$

In particular, at least one of the $q_{i,1}$ is a unit (see the very beginning of the proof.)

Assertion 2. If $q_{i,1}$ is a unit, then so is q_i.

Proof. As the assertion is clear if $\tilde{q}_i$ is not a unit, we can assume that $\tilde{q}_i$ is a unit. Let χ_i denote the vector field $q_{i,1}^{-1}Q_i'$; in particular $\chi_i(h_1) = h_1$. As h_1 defines an isolated singularity, a famous result due to K. Saito [19] asserts that, up to a change of coordinates, χ_i is an Euler vector field $\sum_{k=1}^{n} \alpha_k x_k \partial_k$ with $\alpha_k \in \mathbf{Q}^{*+}$. Hence, the relation $\chi_i(\tilde{h}) = q_{i,1}^{-1}\tilde{q}_i\tilde{h}$ implies that the constant $(q_{i,1}^{-1}\tilde{q}_i)(0)$ belongs

[3] In fact, the same proof shows directly that condition $\mathbf{A}(1/h)$ implies $\mathbf{B}(1/\tilde{h}, h_1)$.

to $\mathbf{Q}^{*+}$ [consider the initial part of $q_{i,1}^{-1}\tilde{q}_i\tilde{h}$ relative to $\alpha_1, \ldots, \alpha_n$]. In particular, $q_{i,1}^{-1}\tilde{q}_i + 1$ is a unit, and so is $q_i = \tilde{q}_i + q_{i,1}$. □

We recall that a formal power series $g \in \mathbf{C}[[x_1, \ldots, x_n]]$ is *weakly weighted homogeneous* of type $(\beta_0, \beta_1, \ldots, \beta_n) \in \mathbf{C}^{n+1}$ if for all monomial $x_1^{\gamma_1} \cdots x_n^{\gamma_n}$ with a nonzero coefficient in the power expansion of g, we have $\beta_1\gamma_1 + \cdots + \beta_n\gamma_n = \beta_0$. Let us pursue the proof. We have proved that there exists an Euler vector field χ_i such that $q_i^{-1}\chi_i(h) = h$ (in particular, $q_i(0) > 0$). From [19], Corollary 3.3, there exists a formal change of coordinates ϕ such that $h \circ \phi$ is weakly weighted homogeneous of type $(1, \alpha_1 q_i^{-1}(0), \ldots, \alpha_n q_i^{-1}(0))$. As the $\alpha_k q_i^{-1}(0)$ are strictly positive, $h \circ \phi$ is in fact weighted homogeneous, and according to a theorem of Artin [1], a convergent change of coordinates exists. This completes the proof. □

Proof of Theorem 1.3. The case $n = 2$ is done in [26], Theorem 1.2. We just have to check that the condition 2 in the previous statement may be replaced by $\mathbf{B}(h)$. Indeed, condition $\mathbf{A}(1/h)$ always implies $\mathbf{B}(h)$ ([28] Proposition 1.3), and on the other hand, $\mathbf{B}(h)$ is stronger than $\mathbf{B}(1/\tilde{h}, h_1)$ (Proposition 1.1). □

Of course, we can use §2.2 to test if condition $\mathbf{B}(h)$ is verified. In the particular case $p = 2$ and h weighted homogeneous, we obtain the following characterization:

Corollary 3.6 *Let $h_1, h_2 \in \mathbf{C}[x_1, \ldots, x_n]$ be two weighted homogeneous polynomial of degree d_1, d_2 for a system $\alpha \in (\mathbf{Q}^{*+})^n$, defining hypersurfaces with an isolated singularity at the origin and without common components. Let $\mathcal{K} \subset \mathcal{O}$ be the ideal generated by the maximal minors of the Jacobien matrix of (h_1, h_2). Then the annihilator of $1/h_1h_2$ is generated by operators of order 1 if and only if for $j = 1$ or 2, there is no weighted homogeneous element in $\mathcal{O}/h_j\mathcal{O} + \mathcal{K}$ whose weight belongs to the set $\{d_j \times k - \sum_{i=1}^n \alpha_i \ ; \ k \in \mathbf{N} \ \& \ k \geq 2\}$.*

This relies on the existence of closed formulas for $b((1/\tilde{h})h_1^s, s)$ under these assumptions [25].

3.3 About a family of free germs

In this part, we prove Proposition 1.4. As the two parts are quite distinct, we will prove them successively.

Lemma 3.7 *Let $g \in \mathbf{C}\{x_1, x_2\}$ be a nonzero reduced germ of plane curve such that $g(0) = 0$. Then -1 is the only integral root of the Bernstein polynomial of $(x_1 - x_2x_3)g(x_1, x_2)$.*

Proof. As g is a reduced germ of plane curve, $\mathbf{B}(g)$ is verified [30], [21]. Thus, by using Proposition 1.1, the three conditions $\mathbf{B}((x_1 - x_2x_3)g(x_1,x_2))$, $\mathbf{B}(1/x_1 - x_2x_3, g)$ and $\mathbf{B}(\dot{1}/x_1 - x_2x_3, g)$ are equivalent. Let us prove the last one. From Corollary 2.4, we have $b((\dot{1}/x_1 - x_2x_3)g^s, s) = b((g(x_2x_3, x_2))^s, s)$. Let us write $g(x_2x_3, x_2) = x_2^\ell \tilde{g}(x_2, x_3)$ where $\tilde{g} \in \mathbf{C}\{x_2, x_3\} - x_2\mathbf{C}\{x_2, x_3\}$ is reduced and $\ell \in \mathbf{N}^*$. If $\tilde{g}$ is a unit, then $\mathbf{B}(g(x_2x_3, x_3))$ is verified and so is $\mathbf{B}((x_1 - x_2x_3)g(x_1,x_2))$. Now we assume that $\tilde{g}$ is not a unit. As it is reduced, $\mathbf{B}(\tilde{g})$ is verified and $\mathbf{B}(\tilde{g}x_2^\ell)$ is equivalent to $\mathbf{B}(1/\tilde{g}, x_2^\ell)$. Using Lemma 2.2, it is easy to check that $\mathbf{B}(1/\tilde{g}, x_2)$ implies $\mathbf{B}(1/\tilde{g}, x_2^\ell)$. Thus we just have to prove $\mathbf{B}(1/\tilde{g}, x_2)$. As condition $\mathbf{B}(\tilde{g})$ is verified, the conditions $\mathbf{B}(1/\tilde{g}, x_2)$, $\mathbf{B}(\tilde{g}x_2)$ and $\mathbf{B}(\dot{1}/x_2, \tilde{g})$ are equivalent (Proposition 1.1). Both of them are verified since $b((\dot{1}/x_2)\tilde{g}^s, s) = b((\tilde{g}(0, x_3))^s, s)$ from Corollary 2.4, where $\tilde{g}(0, x_3) = ux_3^N$ with $u \in \mathbf{C}\{x_3\}$ is a unit. This completes the proof. $\square$

We recall that a nonzero germ $h \in \mathcal{O}$ defines a germ of *free* divisor if the module of logarithmic derivations relative to h is $\mathcal{O}$-free [20]. Moreover, such a germ defines a *Koszul-free* divisor if there exists a basis $\{\delta_1, \ldots, \delta_n\}$ of $\mathrm{Der}(-\log h)$ such that the sequence of principal symbols $(\sigma(\delta_1), \ldots, \sigma(\delta_n))$ is $\mathrm{gr}^F\mathcal{D}$-regular.

Lemma 3.8 *Let $g \in \mathbf{C}[x_1, x_2]$ be a weighted homogeneous and reduced polynomial whose multiplicity is greater or equal to 3. Let $h \in \mathbf{C}[x_1, x_2, x_3]$ denote the polynomial $(x_1 - x_2x_3)g(x_1, x_2)$.*

(i) The polynomial h defines a free divisor and verifies the condition $\mathbf{H}(h)$.

(ii) The polynomial h defines a Koszul-free divisor if and only if the weighted homogeneous polynomial g is not homogeneous.

Proof. (i) It is enough to remark that the following vector fields verify Saito's criterion [20] for h:

$$\begin{aligned}
\delta_1 &= \alpha_1 x_1 \partial_1 + \alpha_2 x_2 \partial_2 + (\alpha_1 - \alpha_2) x_3 \partial_3 \\
\delta_2 &= g'_{x_2} \partial_1 - g'_{x_1} \partial_2 + (x_3 u - v) \partial_3 \\
\delta_3 &= (x_1 - x_2 x_3) \partial_3
\end{aligned}$$

where $(\alpha_1, \alpha_2) \in (\mathbf{Q}^{*+})^2$ is a system of weights for g, and $u \in \mathbf{C}[x_1, x_2, x_3]$, $v \in \mathbf{C}[x_2, x_3]$ are the polynomials of degree in x_3 less or equal to 1 uniquely defined by the relation

$$x_3 g'_{x_1}(x_1, x_2) + g'_{x_2}(x_1, x_2) = u(x_1, x_2, x_3)x_1 - v(x_2, x_3)x_2$$

(we use that $g'_{x_1}, g'_{x_2} \in (x_1, x_2)\mathbf{C}[x_1, x_2]$ under our assumptions.)

(ii) As the sequence $(\sigma(\delta_1), \sigma(\delta_2), \xi_3)$ is regular, the germ h is Koszul-free if and only if the sequence $(\sigma(\delta_1), \sigma(\delta_2), x_1 - x_2x_3)$ is $\mathcal{O}[\xi]$-regular. By division by $x_1 - x_2x_3$, this condition may be rewritten: *the polynomials*

$$\begin{array}{rcl} \Upsilon_1 & = & \alpha_1 x_2 x_3 \xi_1 + \alpha_2 x_2 \xi_2 + (\alpha_1 - \alpha_2) x_3 \xi_3 \\ \Upsilon_2 & = & g'_{x_2}(x_2x_3, x_2)\xi_1 - g'_{x_1}(x_2x_3, x_2)\xi_2 + (x_3 u(x_2x_3, x_2, x_3) - v(x_2, x_3))\xi_3 \end{array}$$

have no common factor. Let us notice that x_2 is the only (irreducible) common factor of $g'_{x_1}(x_2x_3, x_2)$ and $g'_{x_2}(x_2x_3, x_2)$ [since $g \in \mathbf{C}[x_1, x_2]$ defines an isolated singularity.] Thus, when Υ_1 and Υ_2 have a common factor, this factor is x_2 (up to a multiplicative constant). As g belongs in $(x_1, x_2)^3\mathbf{C}[x_1, x_2]$, we have $g'_{x_1}, g'_{x_2} \in (x_1, x_2)^2\mathbf{C}[x_1, x_2]$; thus $u, v \in (x_1, x_2)\mathbf{C}[x_1, x_2, x_3]$. In particular, x_2 is a factor of Υ_2, and Υ_1, Υ_2 have no common factor if and only if $\alpha_1 \neq \alpha_2$. This completes the proof. $\square$

Of course, for $g = x_1x_2(x_1 + x_2)$, h is the example of F.J. Calderón-Moreno in [4] and it is not Koszul-free.

Proof of Proposition 1.4, part (i). Without loss of generality, we will assume that $\delta_1(h) = h$. Let us take $\delta'_2 = \delta_2 - u \cdot \delta_1$ and $\delta'_3 = \delta_3 + x_2\delta_1$; in particular, $\{\delta_1, \delta'_2, \delta'_3\}$ is a basis of Der(log h) such that $\delta'_2(h) = \delta'_3(h) = 0$.

From the characterization of condition $\mathbf{A}(1/h)$ for Koszul-free germs (see [28] Corollary 1.8), it is enough to check that condition $\mathbf{A}(h)$ fails, that is, the sequence $(x_1 - x_2x_3, \sigma(\delta'_2), \sigma(\delta'_3))$ is not regular. As g belongs to $(x_1, x_2)^3\mathbf{C}[x_1, x_2]$, we have $\sigma(\delta'_2), \sigma(\delta'_3) \in (x_1, x_2)\mathcal{O}[\xi]$. By division by $x_1 - x_2x_3$, we deduce that the sequence is not regular. $\square$

Notation 3.9 Given a homogeneous polynomial $g \in \mathbf{C}[x_1, x_2] - \mathbf{C}$ of degree $p \geq 1$, we denote by $\tilde{g}_1, \tilde{g}_2 \in \mathbf{C}[x_1, x_2, x_3]$ the quotient of the division of g'_{x_1}, g'_{x_2} by $x_1 - x_2x_3$. In particular:

$$g'_{x_i} = (x_1 - x_2x_3)\tilde{g}_i + x_2^{p-1} g'_{x_i}(x_3, 1), \; i \in \{1, 2\}. \tag{5}$$

Lemma 3.10 *Let $g \in \mathbf{C}[x_1, x_2]$ be a homogeneous reduced polynomial of degree $p \geq 3$. Then the characteristic variety of $\mathcal{D}(1/x_1 - x_2x_3)g^s$ is defined by the following polynomials: $(x_1 - x_2x_3)\xi_3$, $g'_{x_2}\xi_1 - g'_{x_1}\xi_2 + p x_2^{p-2} g(x_3, 1)\xi_3$, and $[x_2 g'_{x_2}(x_3, 1)\xi_1 - x_2 g'_{x_1}(x_3, 1)\xi_2 + p g(x_3, 1)\xi_3]\xi_3$.*

Proof. Using [14] Proposition 2.14.4, the characteristic variety of the $\mathcal{D}$-module $\mathcal{D}(1/x_1 - x_2x_3)g^s$ is the union of the conormal spaces W_g and $W_{g|x_1 = x_2x_3}$. It is easy to check that they are defined by the ideals $I_1 = (\xi_3, g'_{x_2}\xi_1 - g'_{x_1}\xi_2)\mathcal{O}[\xi]$ and $I_2 = (x_1 - x_2x_3, x_2 g'_{x_2}(x_3, 1)\xi_1 - x_2 g'_{x_1}(x_3, 1)\xi_2 + p g(x_3, 1)\xi_3)\mathcal{O}[\xi]$ respectively. Clearly, the ideal I generated by the given polynomials is contained in $I_1 \cap I_2$. Thus we just have to prove the reverse relation.

Let $A, B, C, D \in \mathcal{O}[\xi]$ be such that

$$A(x_1 - x_2x_3) + B(x_2 g'_{x_2}(x_3, 1)\xi_1 - x_2 g'_{x_1}(x_3, 1)\xi_2 + p g(x_3, 1)\xi_3) = C\xi_3 + D(g'_{x_2}\xi_1 - g'_{x_1}\xi_2).$$

Using (5), we get

$$(A - D(\tilde{g}_2\xi_1 - \tilde{g}_1\xi_2))(x_1 - x_2x_3) + (pBg(x_3,1) - C)\xi_3$$
$$+ (B - Dx_2^{p-2})x_2(g'_{x_2}(x_3,1)\xi_1 - g'_{x_1}(x_3,1)\xi_2) = 0$$

Since the sequence $(x_1 - x_2x_3, \xi_3, x_2(g'_{x_2}(x_3,1)\xi_1 - g'_{x_1}(x_3,1)\xi_2))$ is $\mathcal{O}[\xi]$-regular, there exist $U, V, W \in \mathcal{O}[\xi]$ such that

$$\begin{cases} A - D(\tilde{g}_2\xi_1 - \tilde{g}_1\xi_2) &= U\xi_3 + Wx_2(g'_{x_2}(x_3,1)\xi_1 - g'_{x_1}(x_3,1)\xi_2) \\ B - Dx_2^{p-2} &= -V\xi_3 - W(x_1 - x_2x_3) \end{cases}$$

Thus one can notice that the first part of the first identity belongs to I, that is, I is the defining ideal of $W_g \cup W_{g|x_1=x_2x_3}$. $\square$

LEMMA 3.11 *Let $g \in \mathbf{C}[x_1, x_2]$ be a homogeneous reduced polynomial of degree 3. Then the annihilator of $(1/x_1 - x_2x_3)g^s$ is generated by the following differential operators:*

$$(x_1 - x_2x_3)\partial_3 - x_2\,,\ \ g'_{x_2}\partial_1 - g'_{x_1}\partial_2 + 3x_2g(x_3,1)\partial_3 + x_3\tilde{g}_1 + \tilde{g}_2 \ \ \textit{and}$$

$$[x_2g'_{x_2}(x_3,1)\partial_1 - x_2g'_{x_1}(x_3,1)\partial_2 + 3g(x_3,1)\partial_3]\partial_3 + \tilde{g}_2\partial_1 - \tilde{g}_1\partial_2 + 3g'_{x_1}(x_3,1)\partial_3 + u'_{x_1}$$

where $u = x_3\tilde{g}_1 + \tilde{g}_2$.

Proof. Let us denote by $I \subset \mathcal{D}$ the ideal generated by the given operators S_1, S_2, S_3. It is not hard to check the inclusion $I \subset \mathrm{Ann}_{\mathcal{D}}\,(1/x_1 - x_2x_3)g^s$. Let us prove that the reverse inclusion by induction on the order of operators.

Let $P \in \mathrm{Ann}_{\mathcal{D}}\,(1/x_1 - x_2x_3)g^s$ be an operator of order d. As $d = 0$ implies $P = 0$, we can assume $d \geq 1$. Then $\sigma(P)$ is zero on the characteristic variety of $\mathcal{D}(1/x_1 - x_2x_3)g^s$. From the previous result, there exists $A_1 \in \mathcal{O}[\xi]$ (resp. A_2, A_3) zero or homogeneous in ξ of degree $d-1$ (resp. $d-1$, $d-2$) such that: $\sigma(P) = \sum_{i=1}^{3} A_i\sigma(S_i)$. If $\tilde{A}_i \in \mathcal{D}$, $1 \leq i \leq 3$, are such that $\sigma(\tilde{A}_i) = A_i$ for $1 \leq i \leq 3$, then $P - \sum_{i=1}^{3}\tilde{A}_iS_i$ belongs to $F_{d-1}\mathcal{D}$ and annihilates $(1/x_1 - x_2x_3)g^s$. By induction, it belongs to I and so does P. $\square$

Proof of Proposition 1.4, part (ii). We will prove that $\mathrm{Ann}_{\mathcal{D}}\,1/h$ is generated by the operators $\tilde{\delta}_1 = \delta_1 + 4$, $\tilde{\delta}_2 = \delta_2 + u$, $\tilde{\delta}_3 = \delta_3 - x_2$ (with the notations introduced in the proof of Lemma 3.8 with $\alpha_1 = \alpha_2 = 1$). From Lemma 3.7, we know that -1 is the smallest integral root of $b((1/x_1 - x_2x_3)g^s, s)$. Thus we have the decomposition $\mathrm{Ann}_{\mathcal{D}}\,1/h = \mathcal{D}\tilde{\delta}_1 + \mathrm{Ann}_{\mathcal{D}}\,(1/x_1 - x_2x_3)g^s$, and the assertion is a direct consequence of the previous result and of the following relation in $\mathcal{D}$:

$$[g'_{x_2}(x_3,1)x_2\partial_1 - g'_{x_1}(x_3,1)x_2\partial_2 + 3g(x_3,1)\partial_3 + 3g'_{x_1}(x_3,1)](\partial_3\tilde{\delta}_1 - \partial_1\tilde{\delta}_3)$$

$$+\ [\partial_2 + x_3\partial_1](\partial_3\tilde{\tilde{\delta}}_2 + (\tilde{g}_2\partial_1 - \tilde{g}_1\partial_2)\tilde{\delta}_3) \;=\; -2S_3 + \partial_1\tilde{\tilde{\delta}}_2 - (\tilde{g}_2\partial_1 - \tilde{g}_1\partial_2 + u'_{x_1})\tilde{\tilde{\delta}}_1$$

where S_3 is the operator of order 2 which appears in the given system of generators of $\mathrm{Ann}_{\mathcal{D}}\,(1/x_1 - x_2x_3)g^s$. □

4 Some other conditions

In this part, $h \in \mathcal{O}$ denotes a nonzero germ such that $h(0) = 0$.

4.1 The condition $\mathbf{A}(h)$ for Sebastiani-Thom germs

We recall that the condition $\mathbf{A}(h)$ on the ideal $\mathrm{Ann}_{\mathcal{D}}\,h^s$ may be considered almost as a geometric condition. Indeed the following condition implies $\mathbf{A}(h)$:

$\mathbf{W}(h)$: The relative conormal space W_h is defined by linear equations in ξ.

since $W_h = \overline{\{(x, \lambda dh)\,|\,\lambda \in \mathbf{C}\}} \subset T^*\mathbf{C}^n$ is the characteristic variety of $\mathcal{D}h^s$ ([16]). For example, $\mathbf{W}(h)$ is true for hypersurfaces with an isolated singularity [32] and for locally weighted homogeneous free divisors [6]. This condition also means that the kernel of the morphism of graded $\mathcal{O}$-algebras:

$$\begin{array}{rcl} \mathcal{O}[X_1, \ldots, X_n] & \longrightarrow & \mathcal{R}(\mathcal{J}_h) \\ X_i & \longmapsto & th'_{x_i} \end{array}$$

is generated by homogeneous elements of degree 1, where $\mathcal{J}_h$ denotes the Jacobian ideal $(h'_{x_1}, \ldots, h'_{x_n})\mathcal{O}$ and $\mathcal{R}(\mathcal{J}_h)$ is the Rees algebra $\bigoplus_{d\geq 0} \mathcal{J}_h^d t^d$. Following a terminology due to W.V. Vasconcelos, one says that $\mathcal{J}_h$ is *of linear type* (see [6] for more details). Finally, let us give a third condition trapped between $\mathbf{A}(h)$ and $\mathbf{W}(h)$:

$\mathbf{G}(h)$: The graded ideal $\mathrm{gr}^F\mathrm{Ann}_{\mathcal{D}}\,h^s$ is generated by homogeneous polynomials in ξ of degree 1.

Remark 4.1 (i) We do not know if the conditions $\mathbf{A}(h)$, $\mathbf{G}(h)$ and $\mathbf{W}(h)$ are - or not - equivalent.
(ii) These conditions are not stable by multiplication by a unit.

It seems uneasy to find sufficient conditions on h for $\mathbf{A}(h)$ or $\mathbf{W}(h)$. Thus, it is natural to study if the class of germs h which verify $\mathbf{A}(h)$ or $\mathbf{W}(h)$ is - or not - stable by Thom-Sebastiani sums. Here we give a positive answer in a particular case.

Proposition 4.2 *Let $g \in \mathcal{O}$ be a nonzero germ such that $g(0) = 0$ and which verifies the condition $\mathbf{W}(g)$. Let $f \in \mathbf{C}\{z_1, \ldots, z_p\}$ be a nonzero germ which defines an isolated singularity at the origin. Then $h = g + f$ verifies the condition $\mathbf{W}(h)$.*

This is direct consequence of the following result.

Proposition 4.3 *Let $g \in \mathcal{O}$ be a nonzero germ such that $g(0) = 0$, and $\Upsilon_1, \dots, \Upsilon_w \in \mathcal{O}[\xi]$ be homogeneous polynomials which generate the defining ideal of W_g.*

Let $f \in \mathbf{C}\{z_1, \dots, z_p\}$ be a nonzero germ which defines an isolated singularity and $\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_p$ denote the conormal coordinates on $T^*\mathbf{C}^n \times \mathbf{C}^p$. Then the relative conormal space $W_{g+f} \subset T^*\mathbf{C}^n \times \mathbf{C}^p$ is defined by the polynomials $f'_{z_i}\eta_j - f'_{z_j}\eta_i$, $1 \leq i < j \leq p$, $g'_{x_k}\eta_i - f'_{z_i}\xi_k$, $1 \leq i \leq p$, $1 \leq k \leq n$, and $\Upsilon_1, \dots, \Upsilon_w$.

Proof. Let us denote by $E \subset \mathbf{C}\{z_1, \dots, z_p\}$ a $\mathbf{C}$-vector space of finite dimension isomorphic to $\mathbf{C}\{z_1, \dots, z_p\}/(f'_{z_1}, \dots, f'_{z_p})$ by projection, and by $\mathbf{C}\{x, z\}$ the ring $\mathbf{C}\{x_1, \dots, x_n, z_1, \dots, z_p\}$. In particular, any germ $p \in \mathbf{C}\{x, z\}$ may be written in a unique way: $p = \tilde{p} + r$ where $\tilde{p} \in E \otimes_{\mathbf{C}} \mathcal{O} \subset \mathbf{C}\{x, z\}$ and $r \in (f'_{z_1}, \dots, f'_{z_p})\mathbf{C}\{x, z\}$.

We denote by $I_{f+g} \subset \mathbf{C}\{x, z\}[\xi, \eta]$ the ideal generated by the given operators, and by $I_g \subset \mathbf{C}\{x, z\}[\xi, \eta]$ (resp. I_f) the ideal generated by $\Upsilon_1, \dots, \Upsilon_w$ (resp. $f'_{z_i}\eta_j - f'_{z_j}\eta_i$, $1 \leq i < j \leq p$). Obviously, any element of I_{g+f} is zero on W_{g+f}. Let us prove the reverse relation.

Let $P \in \mathbf{C}\{x, z\}[\xi, \eta]$ be a homogeneous polynomial of degree $N \in \mathbf{N}^*$ in (ξ, η) which is zero on W_{g+f}.

Assertion 1. There exists $\tilde{P}(\xi, \eta) \in \mathbf{C}\{x, z\}[\xi, \eta]$ such that $P - \tilde{P}(\xi, \eta)$ belongs to I_{g+f}, and it is of the form:

$$\tilde{P}(\xi, \eta) = Q(\eta) + \sum_{|\gamma| \leq N-1} \tilde{P}_\gamma(\xi)\eta_1^{\gamma_1} \cdots \eta_p^{\gamma_p}$$

where $\gamma = (\gamma_1, \dots, \gamma_p) \in \mathbf{N}^p$, $\tilde{P}_\gamma(\xi) \in (E \otimes \mathcal{O})[\xi]$ are zero or homogeneous in ξ of degree $N - |\gamma|$, $Q(\eta) \in \mathbf{C}\{x, z\}[\eta]$ is zero or homogeneous of degree N.

Proof. Let us write: $P = \sum_{|\beta+\gamma|=N} p_{\beta,\gamma}\eta^\gamma\xi^\beta$ with $p_{\beta,\gamma} \in \mathcal{O}$. For all $\beta \in \mathbf{N}^n$, $|\beta| = N$, the germ $p_{\beta,0}$ may be written in a unique way $p_{\beta,0} = \tilde{p}_{\beta,0} + r_{\beta,0}$ with $\tilde{p}_{\beta,0} \in E \otimes \mathcal{O}$ and $r_{\beta,0} = \sum_{i=1}^p r_{\beta,0,i} f'_{z_i}$ for some $r_{\beta,0,i} \in \mathbf{C}\{x, z\}$. As $|\beta| \geq 1$, there exists an index k such that $\beta_k \neq 0$. Thus

$$r_{\beta,0}\xi_1^{\beta_1} \cdots \xi_n^{\beta_n} - \sum_{i=1}^p r_{\beta,0,i} g'_{x_k}\eta_i\xi_1^{\beta_1} \cdots \xi_k^{\beta_k-1} \cdots \xi_n^{\beta_n} \in I_{g+f}$$

and we fix $\tilde{P}_0(\xi) = \sum_{|\beta|=N} \tilde{p}_{\beta,0}\xi^\beta$. By iterating this process for increasing $|\gamma|$, we get a decomposition $P = Q(\eta) + \sum_{|\gamma| \leq N-1} \tilde{P}_\gamma(\xi)\eta^\gamma + R$ where $R \in I_{g+f}$. $\square$

Assertion 2. The polynomials $\tilde{P}_\gamma(\xi)$ belong to I_g.

Proof. We prove it by induction on γ, using the lexicographical order on $\mathbf{N}^p$. As $\tilde{P}(g'_{x_1},\ldots,g'_{x_n},f'_{z_1},\ldots,f'_{z_p})=0$, we have $\tilde{P}_0(g'_{x_1},\ldots,g'_{x_n})\in(f'_{z_1},\ldots,f'_{z_p})\mathbf{C}\{x,z\}$. Thus $\tilde{P}_0(\xi)$ belongs to I_g (since $\tilde{P}_0(\xi)\in(E\otimes\mathcal{O})[\xi]$ and $g\in\mathcal{O}$). Now, let us assume that $\tilde{P}_{\gamma'}(\xi)\in I_g$ for all $\gamma'<\gamma$, $\gamma'\geq 0$ and $\tilde{P}_\gamma(\xi)\neq 0$. Since $\tilde{P}(g'_{x_1},\ldots,g'_{x_n},f'_{z_1},\ldots,f'_{z_p})=0$ and $\tilde{P}_{\gamma'}(g'_{x_1},\ldots,g'_{x_n})=0$ for $\gamma'<\gamma$, we have:

$$\begin{aligned}\tilde{P}_\gamma(g'_{x_1},\ldots,g'_{x_n}){f'}^{\gamma_1}_{z_1}\cdots{f'}^{\gamma_p}_{z_p} \;\in\;& ({f'}^{\gamma_1+1}_{z_1},{f'}^{\gamma_1}_{z_1}{f'}^{\gamma_2+1}_{z_2},\ldots,{f'}^{\gamma_1}_{z_1}\cdots{f'}^{\gamma_{p-1}}_{z_{p-1}}{f'}^{\gamma_p+1}_{z_p})\mathbf{C}\{x,z\}\\ &+\;Q(f'_{z_1},\ldots,f'_{z_p})\mathbf{C}\{x,z\}\\ &\subset({f'}^{\gamma_1+1}_{z_1},\ldots,{f'}^{\gamma_p+1}_{z_p})\mathbf{C}\{x,z\}\end{aligned}$$

since the degree of $Q(\eta)$ is strictly greater than $|\gamma|$. From this identity, we deduce that $\tilde{P}_\gamma(g'_{x_1},\ldots,g'_{x_n})\in(f'_{z_1},\ldots,f'_{z_p})\mathbf{C}\{x,z\}$ using that $(f'_{z_1},\ldots,f'_{z_p})$ is a $\mathbf{C}\{x,z\}$-regular sequence. Thus $\tilde{P}_\gamma(\xi)$ belongs to I_g as above. □

In particular, the polynomial $P-Q(\eta)$ belongs to I_{g+f}. As P is zero on W_{g+f}, we have $Q(f'_{z_1},\ldots,f'_{z_p})=0$. Thus $Q(\eta)$ belongs to I_f (since $(f'_{z_1},\ldots,f'_{z_p})$ is $\mathbf{C}\{x,z\}$-regular). We conclude that $P\in I_{g+f}$, and this completes the proof. □

Remark 4.4 Let us recall that the reduced Bernstein polynomial of the germ $h=g(x)+z^N$ has no integral root for N 'generic' [21]. In particular, our result allows to construct some examples of weighted homogeneous polynomials h which verify condition $\mathbf{A}(1/h)$ [with the help of identity (1) of the Introduction].

4.2 The condition $\mathbf{A}_{\log}(1/h)$

Let us recall how the condition $\mathbf{A}(1/h)$ appears in the study of the so-called logaritmic comparison theorem. If D is a free divisor, F.J. Calderón-Moreno and L. Narváez-Macarro [8] have obtained a differential analogue of the condition $\mathbf{LCT}(D)$; in particular, it implies that the natural $\mathcal{D}$-linear morphism $\varphi_D:\mathcal{D}_X\otimes_{\mathcal{V}_0^D}\mathcal{O}_X(D)\longrightarrow\mathcal{O}_X(\star D)$ is an isomorphism. Here $\mathcal{O}_X(D)$ denotes the $\mathcal{O}_X$-module of meromorphic functions with at most a simple pole along D, and $\mathcal{V}_0^D\subset\mathcal{D}_X$ is the sheaf of ring of logarithmic differential operators, that is, $P\in\mathcal{D}_X$ such that $P\cdot(h_D)^k\subset(h_D)^k\mathcal{O}$ for any $k\in\mathbf{N}$, where h_D is a (local) defining equation of D. Locally, we have $\mathcal{O}_X(D)=\mathcal{V}_0^D\cdot(1/h_D)$, thus φ_D is given by

$$\begin{array}{rcl}\mathcal{D}/\mathcal{D}\mathrm{Ann}_{\mathcal{V}_0^D}1/h_D & \longrightarrow & \mathcal{O}[1/h_D]\\ P & \longmapsto & P\cdot\dfrac{1}{h_D}\end{array}$$

where $\mathrm{Ann}_{\mathcal{V}_0^D} 1/h_D \subset \mathcal{V}_0^D$ is the ideal of logarithmic operators which annihilate $1/h_D$. From the structure theorem of logarithmic operators associated with a free divisor [4], we have $\mathcal{V}_0^D = \mathcal{O}_X[\mathrm{Der}(-\log h_D)]$; hence the ideal $\mathrm{Ann}_{\mathcal{V}_0^D} 1/h_D$ is locally generated by $v_i + a_i$, $1 \le i \le n$, where $\{v_1, \ldots, v_n\}$ is a basis of $\mathrm{Der}(-\log h_D)$ and $a_i \in \mathcal{O}$ is defined by $v_i(h_D) = a_i h_D$, $1 \le i \le n$. In particular, the injectivity of φ_D means that the condition $\mathbf{A}(1/h)$ is verified.

Let us notice that the following condition may also be considered:

$\mathbf{A}_{\log}(1/h)$ **:** The ideal $\mathrm{Ann}_{\mathcal{D}} 1/h$ is generated by logarithmic operators.

In this paragraph, we compare these two conditions. Firstly, it is easy to see that the condition $\mathbf{A}(1/h)$ always implies $\mathbf{A}_{\log}(1/h)$. On the other hand, we do not know if these conditions are distinct or not. Meanwhile, we have the following result:

Lemma 4.5 *Let $h \in \mathcal{O}$ be a nonzero germ such that $h(0) = 0$. Assume that one of the following conditions is verified:*

1. *the ring $\mathcal{V}_0^D$ coincides with $\mathcal{O}[\mathrm{Der}(-\log h)]$, the $\mathcal{O}$-subalgebra of $\mathcal{D}$ generated by the logarithmic derivations relative to h.*
2. *the conditions $\mathbf{A}(h)$ and $\mathbf{H}(h)$ are verified.*

Then the conditions $\mathbf{A}(1/h)$ and $\mathbf{A}_{\log}(1/h)$ are equivalent.

Proof. Assume that condition 1 is verified, and let $P \in \mathcal{V}_0^D \cap \mathrm{Ann}_{\mathcal{D}} 1/h$ be a nonzero logarithmic operator annihilating $1/h$. By assumption, it may be written as a sum $\sum_{|\gamma| \le d} p_\gamma v_1^{\gamma_1} \cdots v_N^{\gamma_N}$ where $p_\gamma \in \mathcal{O}$ and $v_1, \ldots, v_N$ is a generating system of $\mathrm{Der}(-\log h)$. As $\mathrm{Der}(-\log h)$ is stable by Lie brackets, we have

$$P = \sum_{|\gamma| \le d} p_\gamma (v_1 + a_1)^{\gamma_1} \cdots (v_N + a_N)^{\gamma_N} + \underbrace{\sum_{|\gamma| < d} r_\gamma v_1^{\gamma_1} \cdots v_N^{\gamma_N}}_{R}$$

where $r_\gamma \in \mathcal{O}$, and $a_i \in \mathcal{O}$ is defined by $v_i(h) = a_i h$, $1 \le i \le N$; in particular, R belongs to $\mathcal{V}_0^D \cap \mathrm{Ann}_{\mathcal{D}} 1/h$. By induction, we conclude that P belongs to the ideal $\mathcal{D}(v_1 + a_1, \ldots, v_N + a_N)$; thus $\mathbf{A}_{\log}(1/h)$ implies the condition $\mathbf{A}(1/h)$.

Now we assume that the conditions $\mathbf{A}_{\log}(1/h)$, $\mathbf{A}(h)$ and $\mathbf{H}(h)$ are verified. From Proposition 4.7, the condition $\mathbf{B}(h)$ is also verified. Thus so is $\mathbf{A}(1/h)$ (see (1) in the Introduction). This completes the proof. $\square$

In particular, these conditions coincides for weighted homogeneous polynomials which define an isolated singularity.

Remark 4.6 Some criterions for condition 1 are given by M. Schulze in [23].

Finally, this condition $\mathbf{A}_{\log}(1/h)$ always implies $\mathbf{B}(h)$ (as $\mathbf{A}(1/h)$ does.)

Proposition 4.7 *Let $h \in \mathcal{O}$ be a nonzero germ such that $h(0) = 0$. If the ideal $\operatorname{Ann}_{\mathcal{D}} 1/h$ is generated by logarithmic operators, then -1 is the only integral root of the Bernstein polynomial of h.*

Proof. The proof is analogous to the one of [26], Proposition 1.3. We need the following fact.

Assertion. If Q is a logarithmic operator relative to h, then $Q \cdot h^s = q(s)h^s$ with $q(s) \in \mathcal{O}[s]$.
Proof. We have $Q \cdot h^s = a(s)h^{s-N}$ with $a(s) = \sum_{i=0}^N a_i s^i$, $a_i \in \mathcal{O}$, and N is the degree of Q. Thus we just have to prove that $a(s) \in h^N\mathcal{O}[s]$. As Q is logarithmic, $Q \cdot h^k$ belongs to $h^k\mathcal{O}$ for $k \geq 1$; in particular $\sum_{i=0}^N a_i k^i \in h^N\mathcal{O}$ for $1 \leq k \leq N+1$. By solving this system, we get $a_i \in h^N\mathcal{O}$, $0 \leq i \leq N$, that is, $a(s) \in h^N\mathcal{O}[s]$. □

Let $Q_1, \ldots, Q_w$ be a generating system of logarithmic operators which annihilate $1/h$. For $1 \leq i \leq w$, we have $Q_i \cdot h^s = q_i(s)h^s$ with $q_i(s) \in \mathcal{O}[s]$. As Q_i annihilates $1/h$, the polynomial $q_i(s)$ belongs to $(s+1)\mathcal{O}[s]$ and we denote $\tilde{q}_i(s) \in \mathcal{O}[s]$ the quotient of $q_i(s)$ by $(s+1)$. Let us suppose that the Bernstein polynomial of h, denoted by $b(s)$, has an integral root strictly smaller than -1. We denote by $k \leq -2$, the greatest integral root of $b(s)$ verifying this condition. Using a Bernstein equation which gives $b(s)$, we get:

$$b(s) \cdots b(s-k-2)h^s = P(s)h^{s-k-1}$$

where $P(s) \in \mathcal{D}[s]$. Thus $P(k)$ annihilates $1/h$ and it may be written $\sum_{i=1}^w A_i Q_i$ with $A_i \in \mathcal{D}$, $1 \leq i \leq w$. If $P'(s) \in \mathcal{D}[s]$ is the quotient of $P(s)$ by $s-k$, the previous equation becomes:

$$\underbrace{b(s) \cdots b(s-k-2)}_{c(s)} h^s = (s-k)\left[P'(s) + \sum_{i=1}^w A_i \tilde{q}_i\right] h^{-k-2} \cdot h^{s+1}$$

where $-k-2 \geq 0$ and the multiplicity of k in $c(s)$ is the same in $b(s)$. Hence, by division by $(s-k)$, we get a Bernstein functional equation such that the polynomial in the left member is not a multiple of $b(s)$. But this is not possible, because $b(s)$ is the Bernstein polynomial of h. Hence we have the result. □

4.3 The condition $\mathbf{M}(h)$

Let $h \in \mathcal{O}$ be a nonzero germ such that $h(0) = 0$. In this paragraph, we study the following condition

$\mathbf{M}(h)$**:** The $\mathcal{D}$-module $\widetilde{\mathcal{M}}_h = \mathcal{D}/\tilde{I}_h$ is holonomic

where $\tilde{I}_h \subset \mathcal{D}$ is the left ideal generated by the operators of order 1 which annihilate $1/h$. This condition only depends on the ideal $h\mathcal{O}$ (since the right multiplication by a unit $u \in \mathcal{O}$ induces an isomorphism of $\mathcal{D}$-modules from $\widetilde{\mathcal{M}}_h$ to $\widetilde{\mathcal{M}}_{uh}$).

Let us recall that this condition and this 'logarithmic' $\mathcal{D}$-module - introduced by F.J Castro-Jiménez and J.M. Ucha in [11] - are very natural in this topic. Indeed, the condition $\mathbf{A}(1/h)$ always implies $\mathbf{M}(h)$, since $\mathbf{A}(1/h)$ means that the morphism $\widetilde{\mathcal{M}}_h \to \mathcal{O}[1/h]$ defined by $P \mapsto P \cdot 1/h$ is an isomorphism. Moreover, the condition $\mathbf{LCT}(D)$ needs locally $\mathbf{M}(h_D)$ for a free divisor D (see the beginning of the previous paragraph).

Here, we link the condition $\mathbf{M}(h)$ with some other conditions introduced in this topic (see §4.1). Firstly, let us consider the following one:

$\mathbf{L}(h)$**:** The ideal in $\mathcal{O}_{T^*\mathbf{C}^n}$ generated by $\pi^{-1}\mathrm{Der}(-\log h)$ defines an analytic space of (pure) dimension n

where π denotes the canonical map $T^*\mathbf{C}^n \to \mathbf{C}^n$. In K. Saito's language, one says that the irreducible components of the *logarithmic characteristic variety* are holonomic; moreover, this is equivalent to the local finiteness of the logarithmic stratification associated with h (see [20], §3). For a free germ, this is exactly the notion of Koszul-free germ (see [20]; [3], Proposition 6.3; [6], Corollary 1.9).

Proposition 4.8 *Let $h \in \mathcal{O}$ be a nonzero germ such that $h(0) = 0$.*
(i) The condition $\mathbf{L}(h)$ implies $\mathbf{M}(h)$.
(ii) The condition $\mathbf{A}(h)$ implies $\mathbf{M}(h)$.
(iii) The condition $\mathbf{G}(h)$ implies $\mathbf{L}(h)$.
(iv) If h defines a locally weighted homogeneous divisor, then the condition $\mathbf{L}(h)$ is verified.

Proof. The first point is clear since $\pi^{-1}\mathrm{Der}(-\log h) \subset \mathrm{gr}\,\tilde{I}_h$. Let us prove (ii). By assumption, the ideal $J = \mathrm{Ann}_{\mathcal{D}}\, h^s$ is included $\tilde{I}$. On the other hand, it is obvious that the operators $h\partial_i + h'_{x_i}$, $1 \le i \le n$, belong to $\tilde{I}$. Hence, we have the following inclusion: $\mathrm{gr}^F J + (h\xi_1, \dots, h\xi_n)\mathcal{O}[\xi] \subset \mathrm{gr}^F \tilde{I}$. We notice that

$$\mathrm{gr}^F J + (h\xi_1, \dots, h\xi_n)\mathcal{O}[\xi] = (\mathrm{gr}^F J, h)\mathcal{O}[\xi] \cap (\xi_1, \dots, \xi_n)\mathcal{O}[\xi]$$

since $\mathrm{gr}^F J \subset (\xi_1, \ldots, \xi_n)\mathcal{O}[\xi]$. Thus the characteristic variety of $\widetilde{\mathcal{M}}_h$ is included in $V(\mathrm{gr}^F J, h) \cup V(\xi_1, \ldots, \xi_n) \subset T^*\mathbf{C}^n$. Let us recall that the characteristic variety of $\mathcal{D}h^s$ is the the closure $W_h \subset T^*\mathbf{C}^n$ of the set $\{(x, \lambda dh(x)) \mid \lambda \in \mathbf{C}\}$ [16]; in particular, W_h is irreducible of pure dimension $n+1$. From the principal ideal theorem, $W_h \cap \{h = 0\} = V(\mathrm{gr}^F J, h)$ has a pure dimension n. Hence $\widetilde{\mathcal{M}}_h$ is holonomic.

The proof of (iii) is the very same, since the ideal generated by the principal symbol of the elements in $\mathrm{Der}(-\log h)$ contains $\mathrm{gr}^F J + (h\xi_1, \ldots, h\xi_n)\mathcal{O}[\xi]$.

Let us prove (iv), by induction on dimension. Let $D \subset \mathbf{C}^n$ denote the hypersurface defined by h, and let L be the associated logarithmic characteristic variety. If $n = 2$, then $\mathbf{W}(h)$ is verified and so is $\mathbf{L}(h)$ by (iii). Now, we assume that $n \geq 3$. From Proposition 2.4 in [9], there exists a neighborhood U of the origin such that, for each point $w \in U \cap D$, $w \neq 0$, the germ of pair $(\mathbf{C}^n, D, w)$ is isomorphic to a product $(\mathbf{C}^{n-1} \times \mathbf{C}, D' \times \mathbf{C}, (0,0))$ where D' is a locally weighted homogeneous divisor of dimension $n-2$. Up to this identification, $\mathrm{Der}(-\log h)_w$ is generated by the elements in $\mathrm{Der}(-\log h_{D'})$ and $\partial/\partial z$, where z is the last coordinate on $\mathbf{C}^{n-1} \times \mathbf{C}$; in particular, the induction hypothesis applied to D' implies the result for $\mathbf{C} \times D'$. Hence, the dimension of $L \cap \pi^{-1}(U - \{0\}) = L - T^*_{\{0\}}\mathbf{C}^n$ is n. Let $C \subset L$ be an irreducible component of L. If $\pi(C) = \{0\}$, then C coincides with $T^*_{\{0\}}\mathbf{C}^n$ since $\dim C$ is at most equal to n (see [3], Proposition 1.14 (i)). Now, if $\pi(C)$ is not the origin, then $\dim C = \dim(C - T^*_{\{0\}}\mathbf{C}^n) = \dim(L - T^*_{\{0\}}\mathbf{C}^n) = n$. We conclude that L has dimension n. $\square$

We recall that K. Saito proved that the condition $\mathbf{L}(h)$ is verified for any hyperplane arrangements [20], Example 3.14. The point (iv) may be considered as a generalization of this fact. On the other hand, it generalizes also the fact that locally weighted homogeneous free divisors are Koszul-free [7] (of course, our proof is similar).

The following diagram summarizes the previous relations:

$$
\begin{tikzcd}
 & \mathbf{W}(h) \arrow[d, Rightarrow] & \\
 & \mathbf{G}(h) \arrow[d, Rightarrow] \arrow[ddl, Rightarrow] & \\
 & \mathbf{A}(h) \arrow[dd, Rightarrow] & \mathbf{A}(1/h) \arrow[ddl, Rightarrow] \\
\mathbf{L}(h) \arrow[dr, Rightarrow] & & \\
 & \mathbf{M}(h) &
\end{tikzcd}
$$

Let us notice that the reverse relations are false. Firstly, if h is the germ $(x_1 - x_2x_3)(x_1x_2^2 + x_1^2x_2)$ then $\mathbf{L}(h)$ and $\mathbf{A}(h)$ are not verified but $\mathbf{A}(1/h)$ holds [20], [5], [6], [10], [28]. On the other hand, if $h = (x_1 - x_2x_3)(x_1^3 + x_2^4)$ then it defines a Koszul-free germ (see Lemma 3.8 for instance); in particular, $\mathbf{L}(h)$ is verified where as $\mathbf{A}(h)$ and $\mathbf{A}(1/h)$ fail (see the proof of Proposition 1.4, (i)). Finally, L. Narváez-Macarro and F.J Calderón-Moreno prove in [8] that the free divisor defined by $h = (x_1 - x_2x_3)(x_1^5 + x_2^4 + x_1^4x_2)$ is not of Spencer type[4]. In fact, the condition $\mathbf{M}(h)$ is no more verified, since all elements of a system of generators of $\tilde{I}$ belongs to $\mathcal{D}(x_1, x_2)$, see [8] §5.

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